Structure-aware combinatorial group testing:

a new method for pandemic screening

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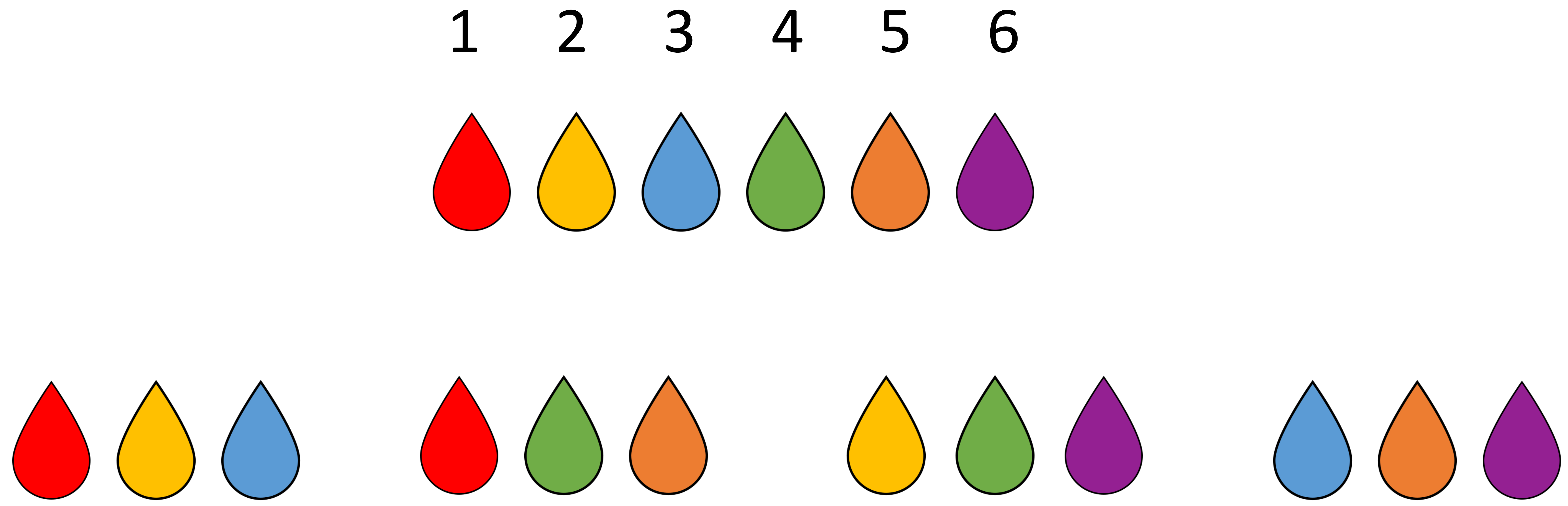
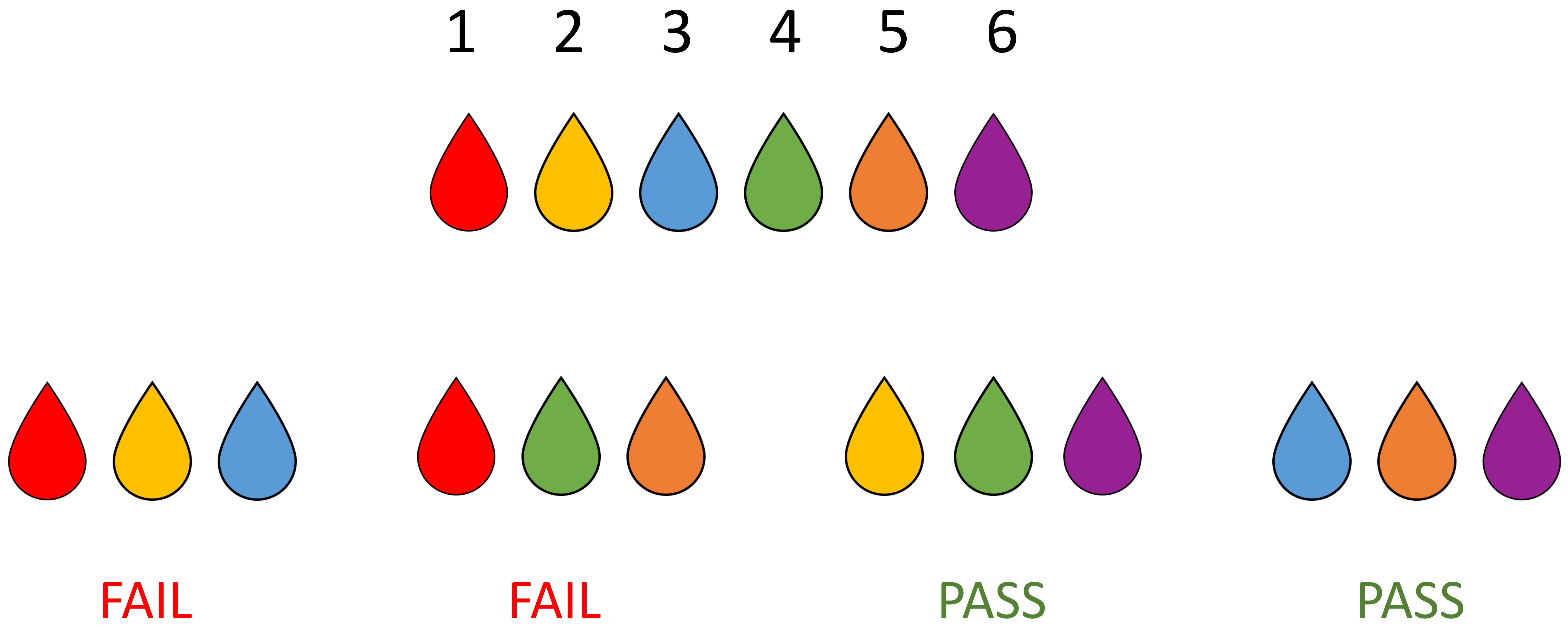
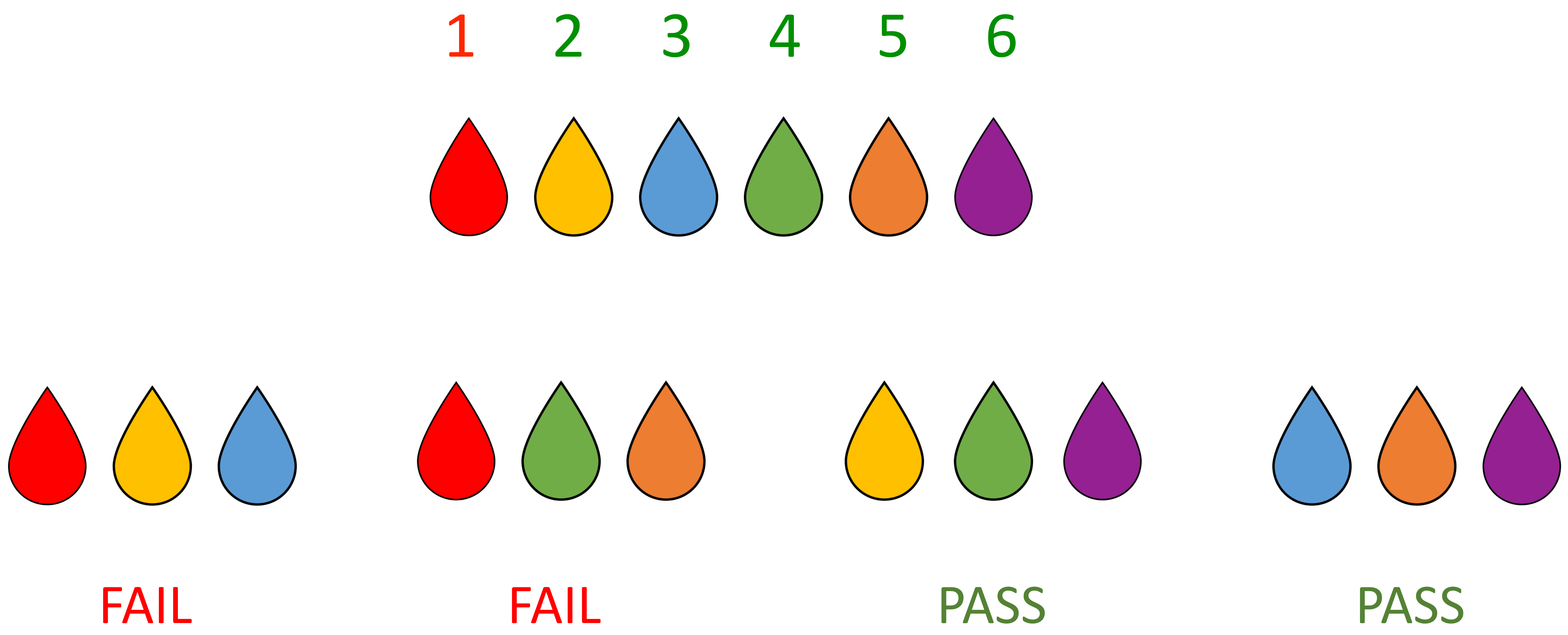
Combinatorial Group Testing



Combinatorial Group Testing

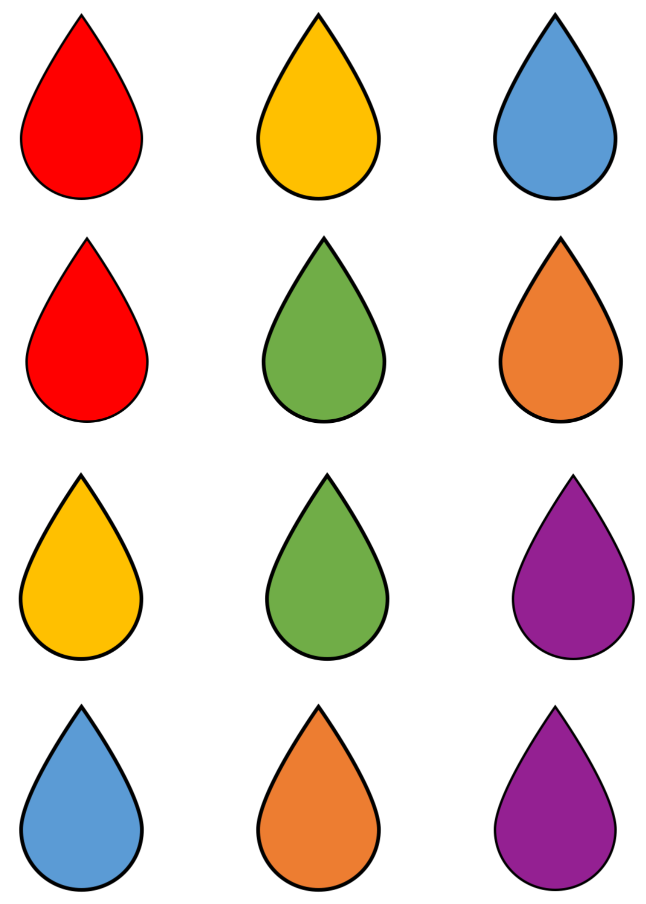


Combinatorial Group Testing



Cover-Free Families

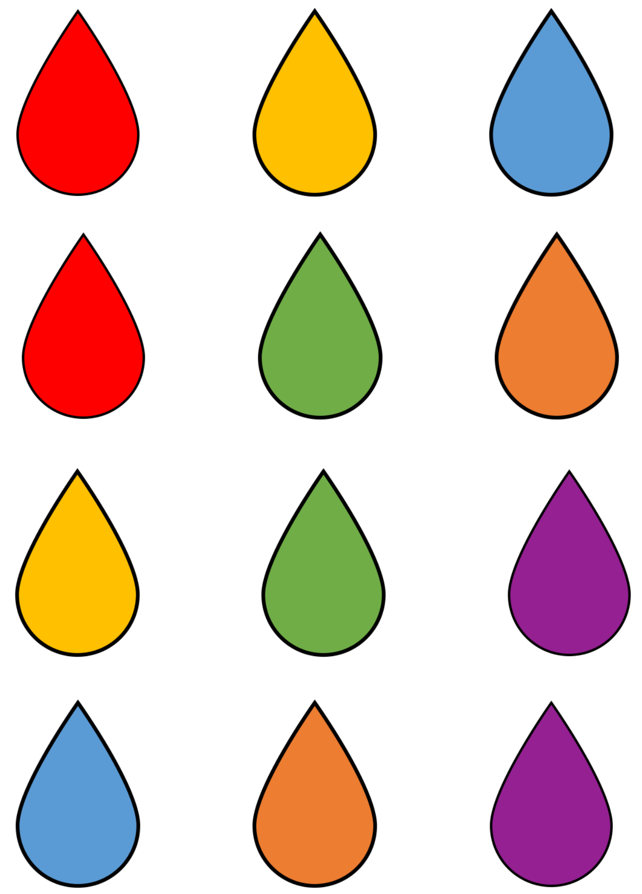
	1	2	3	4	5	6
Test 1	1	1	1	0	0	0
Test 2	1	0	0	1	1	0
Test 3	0	1	0	1	0	1
Test 4	0	0	1	0	1	1



FAIL
FAIL
PASS
PASS

Cover-Free Families

	1	2	3	4	5	6	
Test 1	1	1	1	0	0	0	FAIL
Test 2	1	0	0	1	1	0	FAIL
Test 3	0	1	0	1	0	1	PASS
Test 4	0	0	1	0	1	1	PASS



d – CFF(t, n)

Cover-Free Families

Definition: Let d be a positive integer. A d-cover-free family, denoted $d - CFF(t, n)$, is a set system $\mathcal{F} = (X, \mathcal{B})$ with $|X| = t$ and $|\mathcal{B}| = n$ such that for any $d + 1$ subsets $B_{i_0}, B_{i_1}, \ldots, B_{i_d} \in \mathcal{B}$, we have:

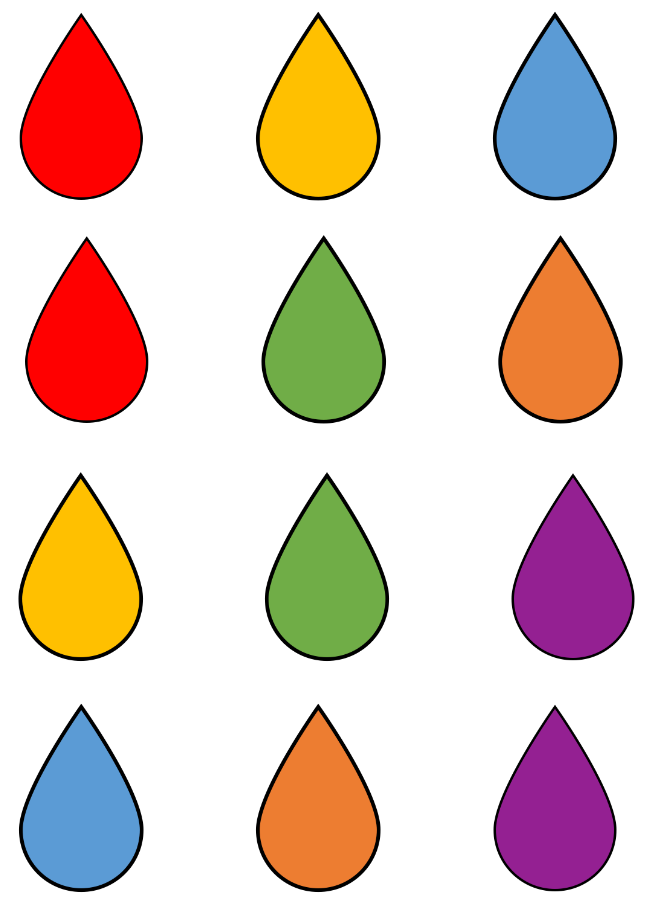
$$\left| B_{i_0} \setminus \left(\bigcup_{j=1}^{d} B_{i_j} \right) \right| \geq 1.$$

No element is ***covered*** by the union of any other d.

* Equivalent to disjunct matrices and superimposed codes.

Cover-Free Families

	1	2	3	4	5	6	
Test 1	1	1	1	0	0	0	FAIL
Test 2	1	0	0	1	1	0	FAIL
Test 3	0	1	0	1	0	1	PASS
Test 4	0	0	1	0	1	1	PASS



***1* – CFF(4, 6)**

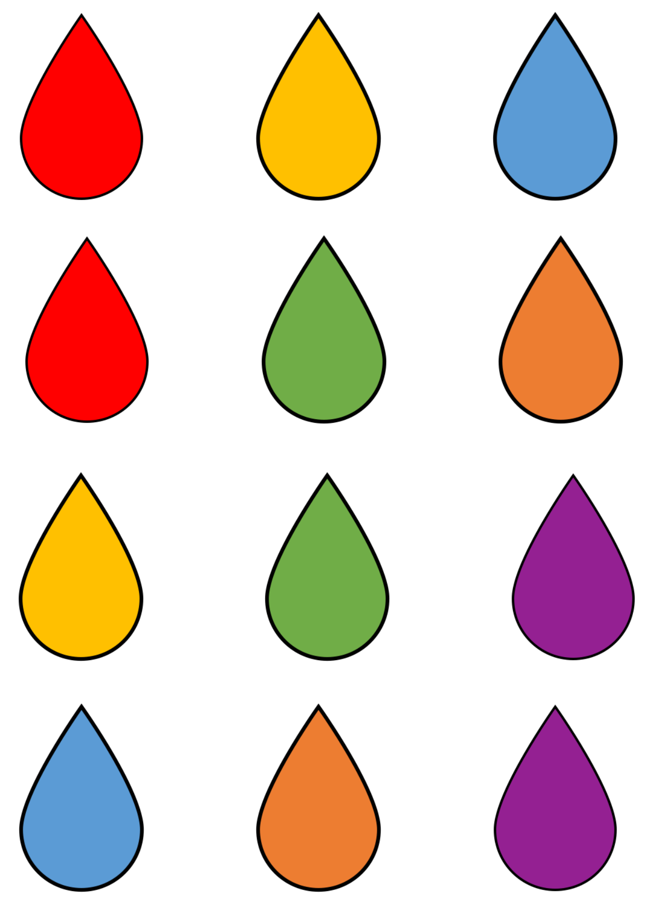
Cover-Free Families

	1	2	3	4	5	6	
Test 1	1	1	1	0	0	0	FAIL
Test 2	1	0	0	1	1	0	FAIL
Test 3	0	1	0	1	0	1	PASS
Test 4	0	0	1	0	1	1	PASS

***1* – CFF(4, 6)**

Cover-Free Families

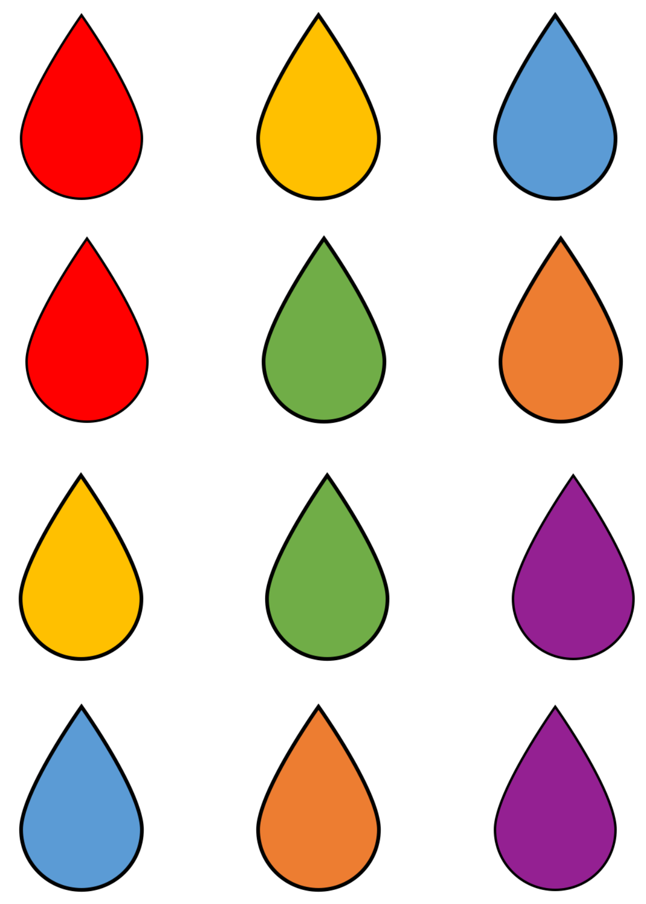
	1	2	3	4	5	6	
Test 1	1	1	1	0	0	0	FAIL
Test 2	1	0	0	1	1	0	FAIL
Test 3	0	1	0	1	0	1	PASS
Test 4	0	0	1	0	1	1	PASS



1 **– CFF(4, 6)**

Cover-Free Families

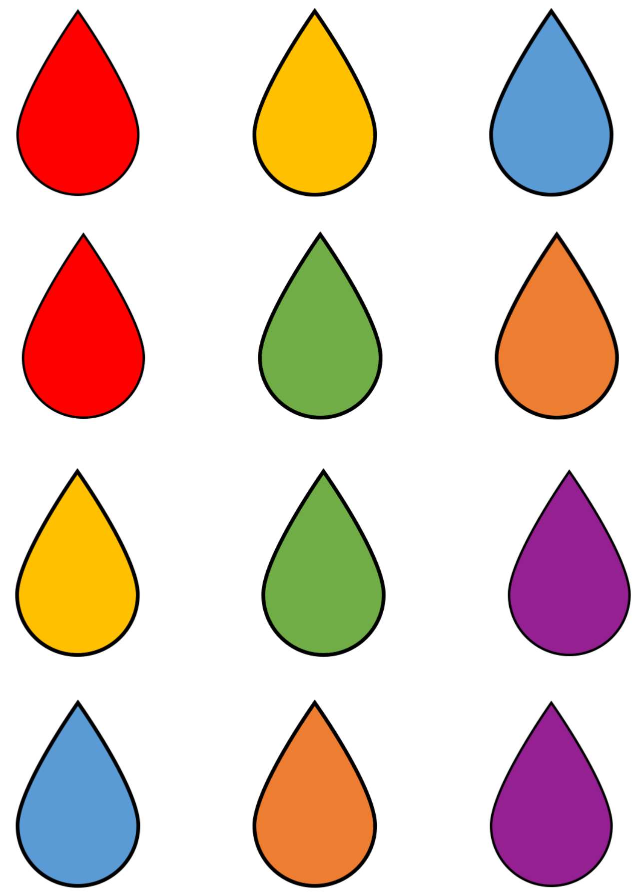
	1	2	3	4	5	6	
Test 1	1	1	1	0	0	0	FAIL
Test 2	1	0	0	1	1	0	FAIL
Test 3	0	1	0	1	0	1	PASS
Test 4	0	0	1	0	1	1	PASS



1 – CFF(4, 6)

Cover-Free Families

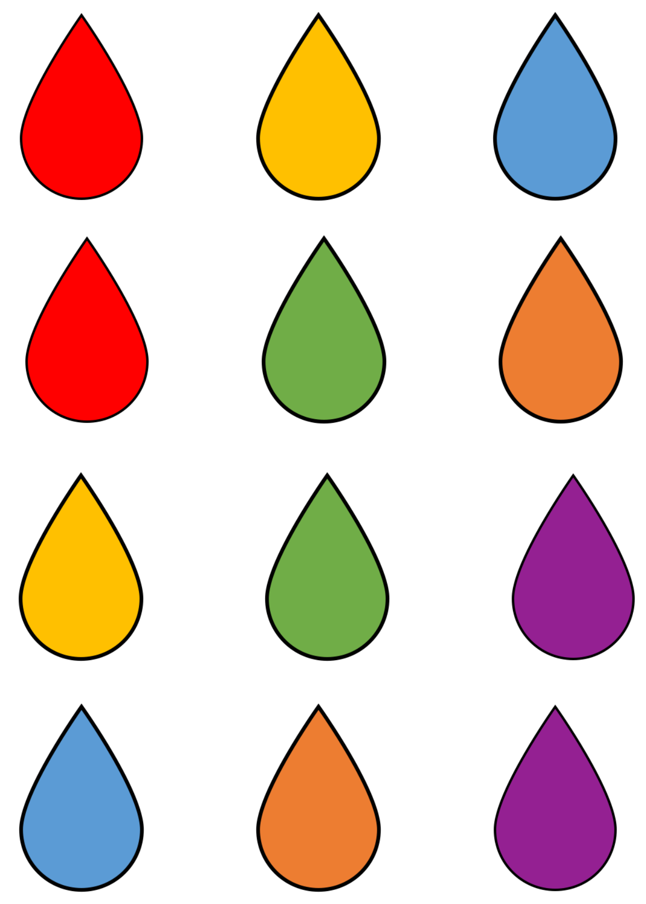
	1	2	3	4	5	6	Result
Test 1	1	1	1	0	0	0	FAIL
Test 2	1	0	0	1	1	0	FAIL
Test 3	0	1	0	1	0	1	PASS
Test 4	0	0	1	0	1	1	PASS



***1* – CFF(4, 6)**

Cover-Free Families

	1	2	3	4	5	6
Test 1	1	1	1	0	0	0
Test 2	1	0	0	1	1	0
Test 3	0	1	0	1	0	1
Test 4	0	0	1	0	1	1



FAIL

FAIL

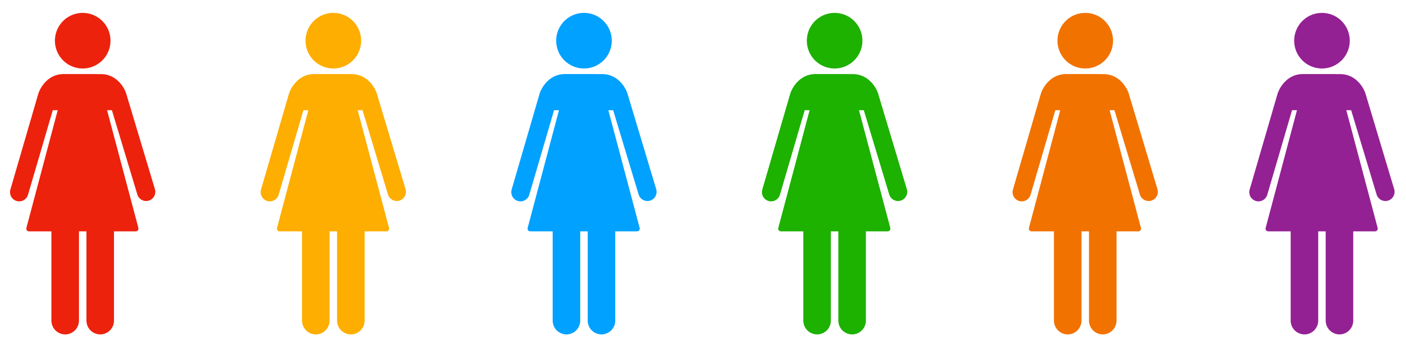
PASS

PASS

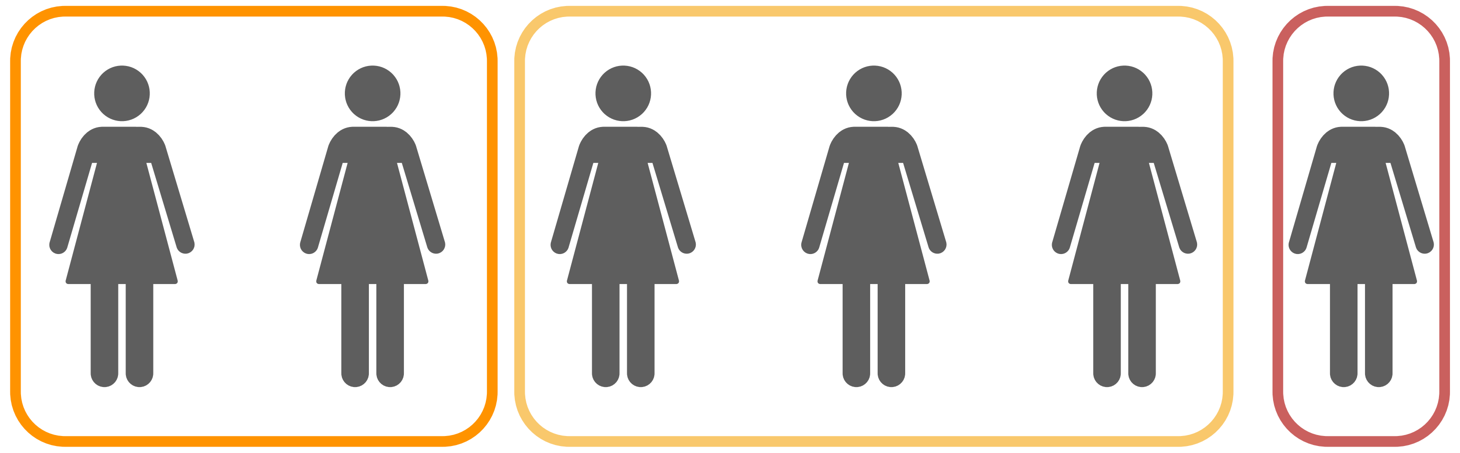
***1* – CFF(4, 6)**

In this talk

1	1	1	0	0	0
1	0	0	1	1	0
0	1	0	1	0	1
0	0	1	0	1	1



- Applications of **combinatorial group testing** in pandemic screening
- Study of ***structure-aware combinatorial group testing*** *
- New constructions of structure-aware CFFs * *
- Future work



* (Nikolopoulos et al., 2021), (Gonen et al., 2022), (My PhD Thesis, 2019)
* * This work

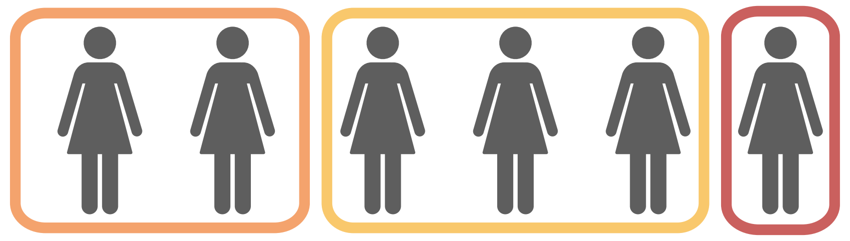
Structure-aware CFFs

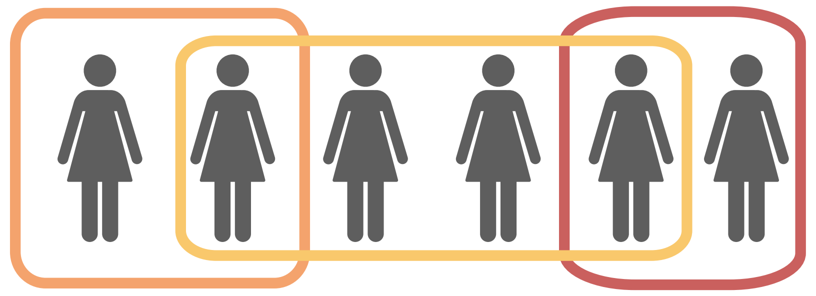
Model the communities as **hypergraphs**

- $\mathscr{H} = (V, \mathscr{S})$

Propose constructions that take $\mathscr{H}$ into consideration

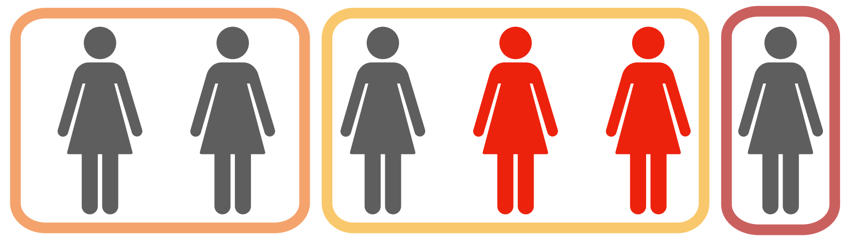
- $(\mathscr{S}, r) - CFF(t, n)$

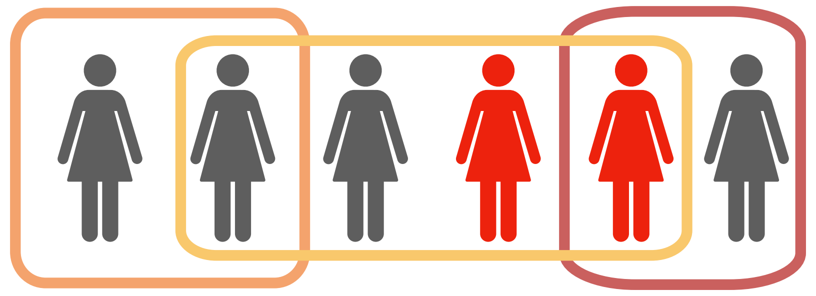




Structure-aware CFFs

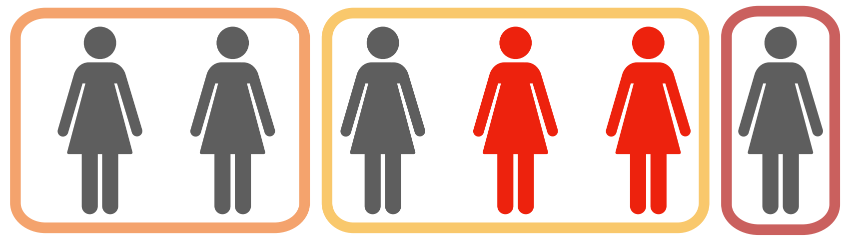
Overlapping and non-overlapping edges:





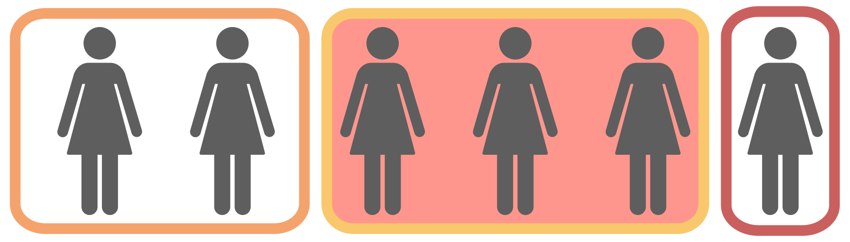
Configurations:

- $(\mathcal{S}, r) - CFF(t, n)$



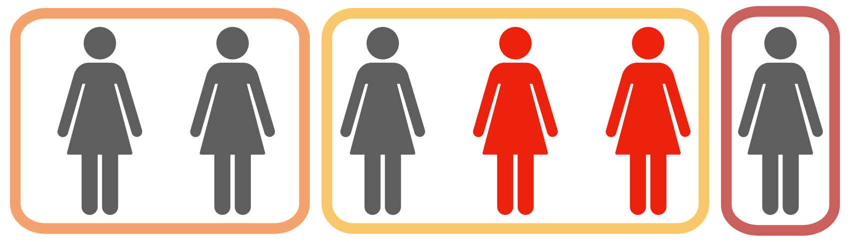
  - Identify all infected individuals, as long as there are at most *r* infected edges that jointly contain them

- $(\mathcal{S}, r) - ECFF(t, n)$



  - Identify *r* infected edges, without internal identification

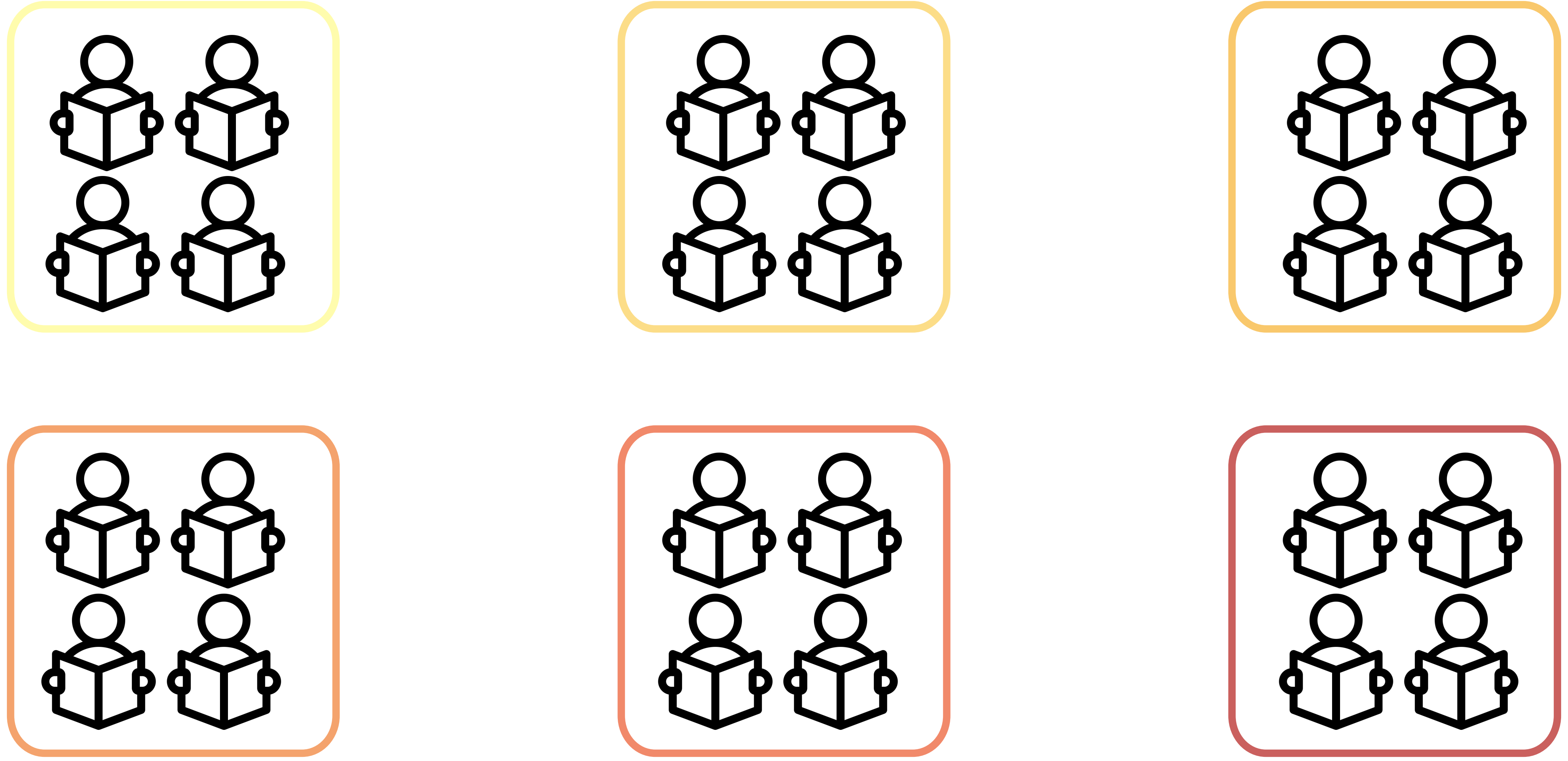
Non-overlapping edges



- Revisit old $d - CFF$ constructions
- Show we can boost the number of infected items they can identify
  - Sperner-type construction for $r = 1$
  - Kronecker-type construction for $r > 1$
  - Array construction
  - Polynomial construction

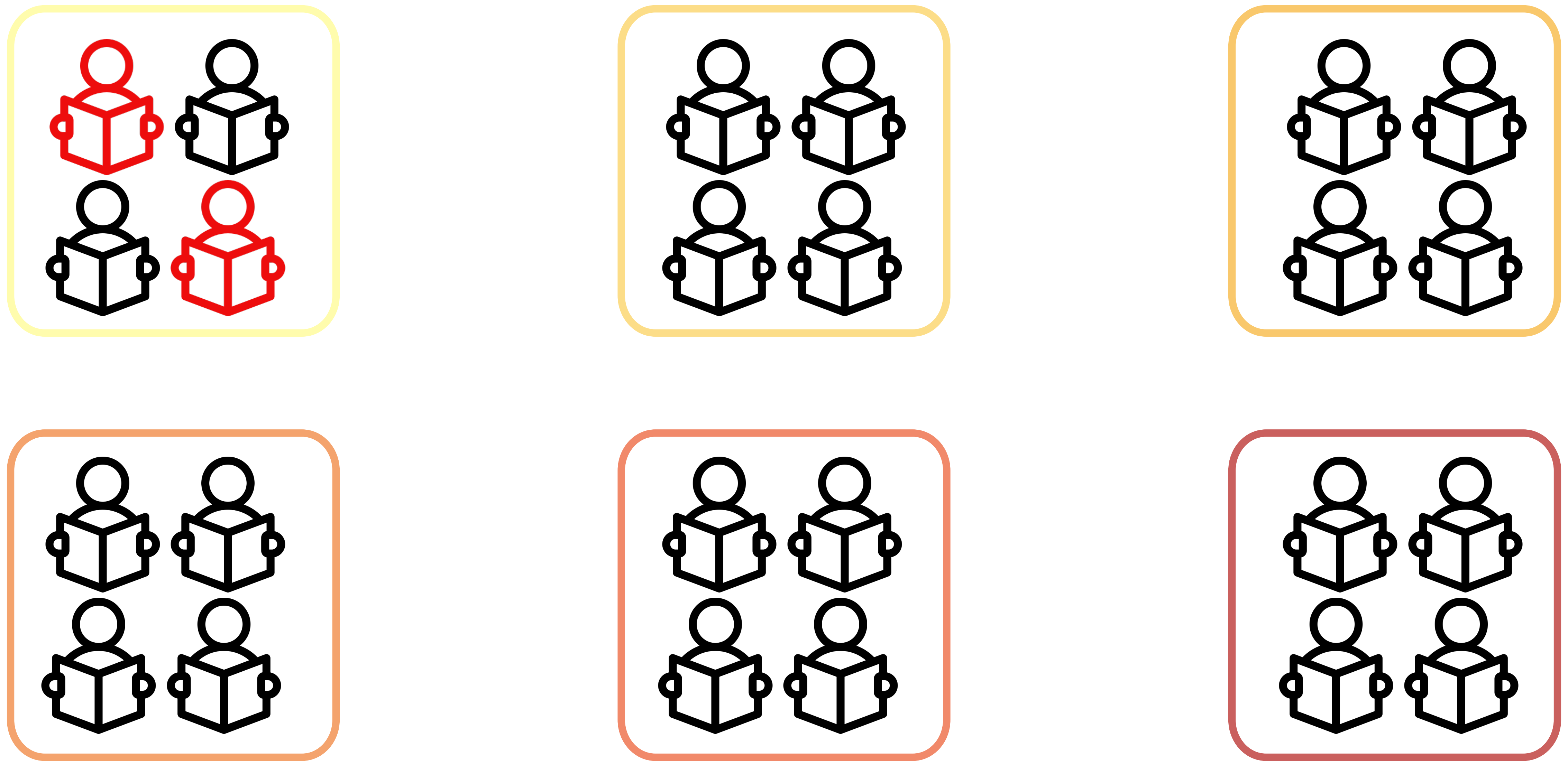
The classroom problem

Non-overlapping edges



The classroom problem

Non-overlapping edges



Sperner-type construction

The classroom problem

One column per classroom

1	1	1	0	0	0
1	0	0	1	1	0
0	1	0	1	0	1
0	0	1	0	1	1

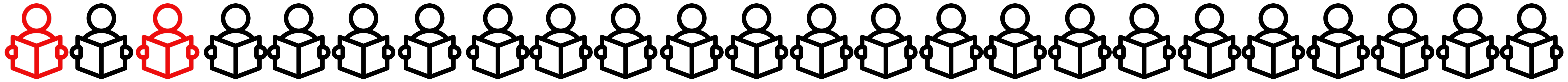
Sperner-type construction

The classroom problem

Classroom 1				Classroom 2				Classroom 3				Classroom 4				Classroom 5				Classroom 6			
1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1

Sperner-type construction

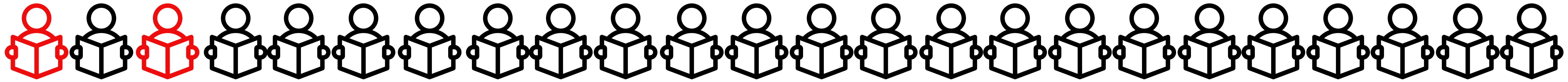
The classroom problem



1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1

Sperner-type construction

The classroom problem

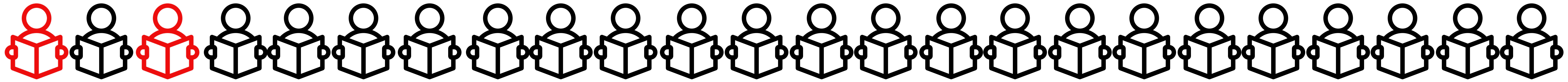


1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	FAIL
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	FAIL
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	PASS
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	PASS

$$(\mathcal{S},1) - ECFF(4,24)$$

Sperner-type construction

The classroom problem

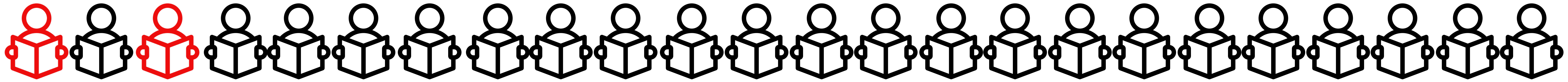


1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1

1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1

Sperner-type construction

The classroom problem



1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	FAIL
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	FAIL
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	PASS
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	PASS
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	FAIL
0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	PASS
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	FAIL
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	PASS

$$(\mathcal{S},1) - CFF(8,24)$$

Sperner-type construction

The classroom problem

- Consider n individuals divided into m non-overlapping edges, each of size up to d.
- Variation of a $1 - CFF(t_1, m)$ concatenated with a $d \times d$ id-matrix.
  - Generates a $(\mathcal{S},1) - CFF(t, n)$, $t = t_1 + d \approx \log m + d = \log n/d + d$
- If we only care about infected edges
  - Restrict to the first t_1 rows to get a $(\mathcal{S},1) - ECFF(t_1, n)$

Sperner-type construction

Comparison with traditional $d-CFF(t,n)$

Total number of students

Number of classrooms

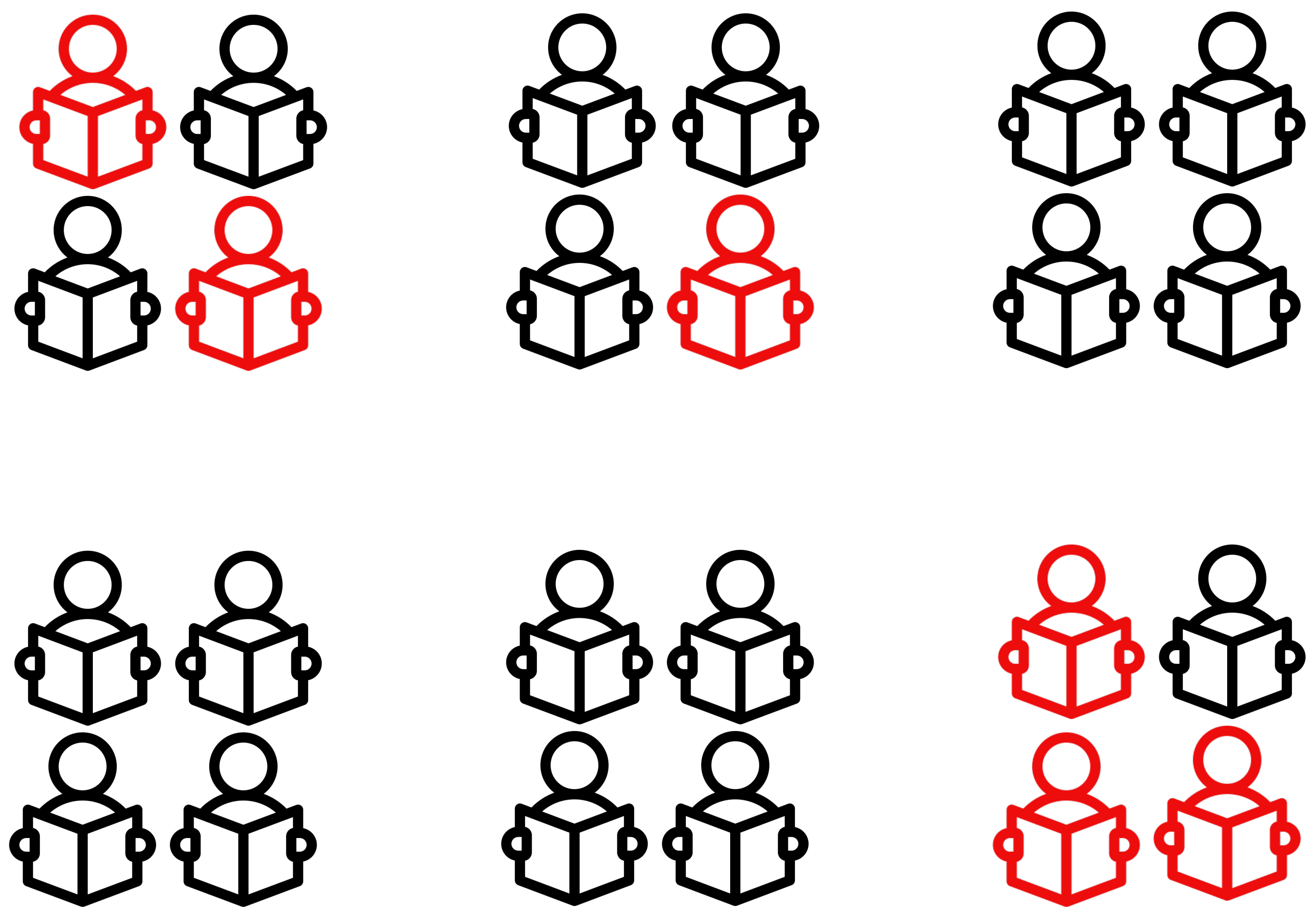
Classroom size

n	m	d	$(\mathcal{S},1)-CFF(t,n)$	$d-CFF(t,n)$
100	10	10	15	66
200	10	20	25	180
300	**10**	**30**	**35**	**231**
100	20	5	11	21
200	20	10	16	66
400	20	20	26	231

Lower bound

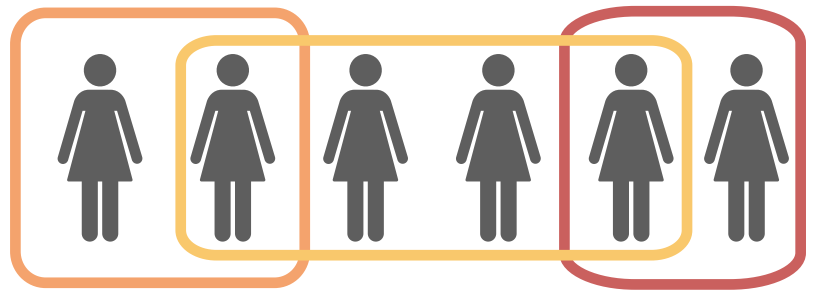
Kronecker-type construction

What if more classrooms are infected?



- Propose some constructions of $(\mathcal{S}, r) - CFF$
  - For m classrooms of k students each
  - Identifies r infected classrooms and everyone inside them
- Generalization of Li, van Rees and Wei (2006)
  - Uses an $r - CFF(t, m)$ to build $(\mathcal{S}, r) - ECFF(t, km)$ and $(\mathcal{S}, r) - CFF(kt, km)$
- Allows edges of different cardinalities

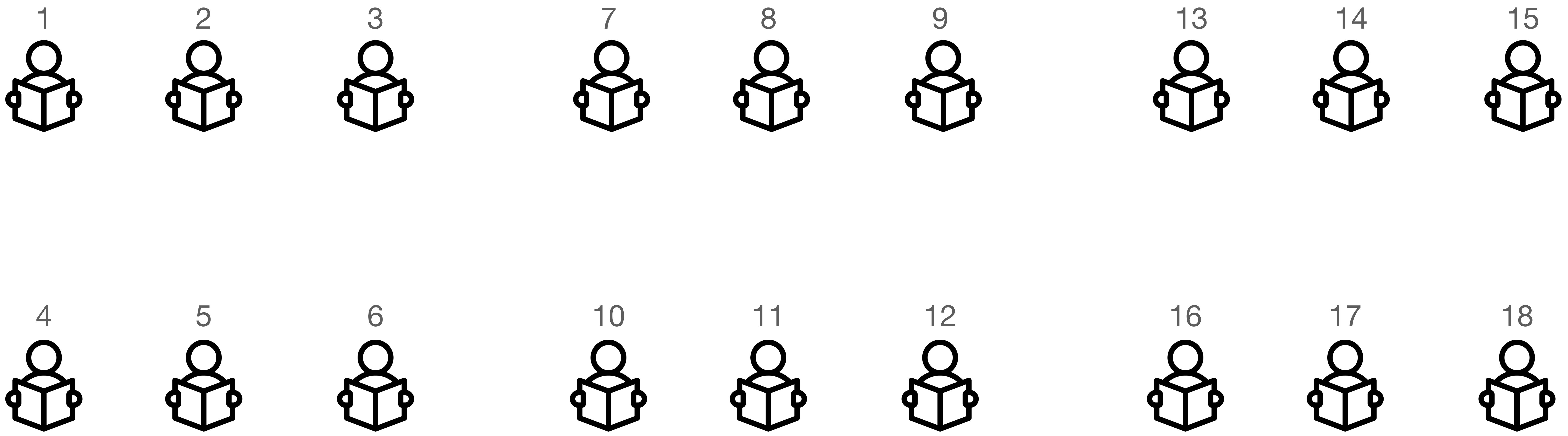
Overlapping edges



- Explore the properties of the hypergraph
- Propose constructions inspired by the non-overlapping ones
  - Construction of $(\mathscr{S},1) - CFF$ and $(\mathscr{S},1) - ECFF$ based on edge-colouring
  - Construction of $(\mathscr{S}, r) - CFF$ based on strong edge-colouring
    - ***Defect cover***: a set of at most r edges whose union contains the set of infected elements

The high school problem

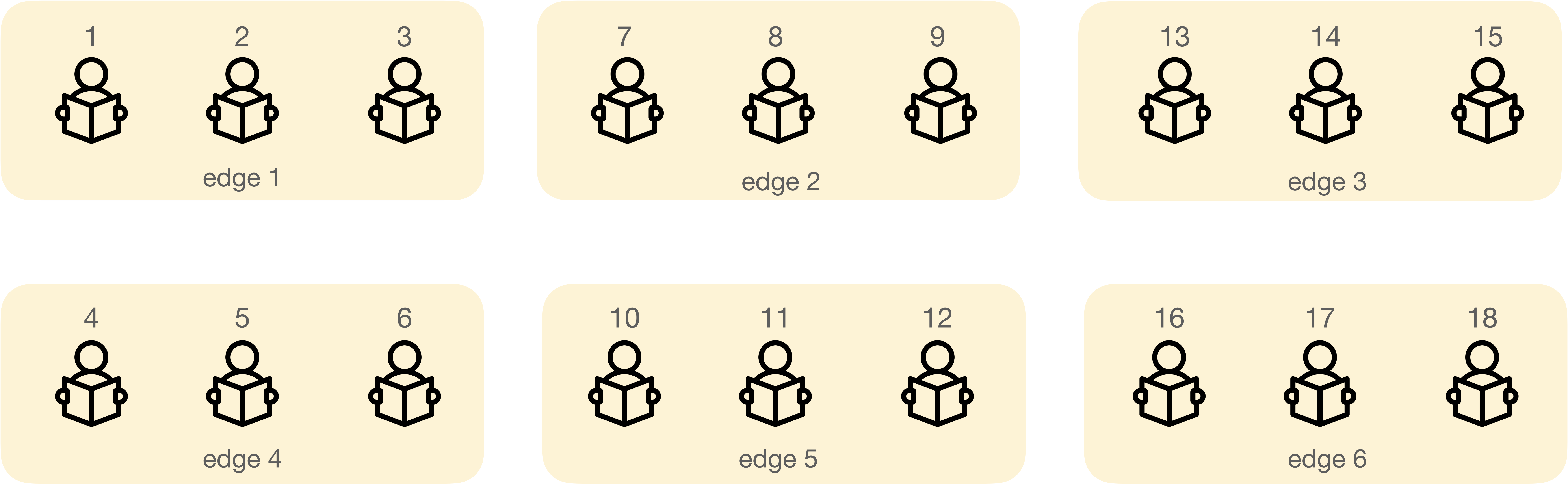
Constructions



The high school problem

Construction

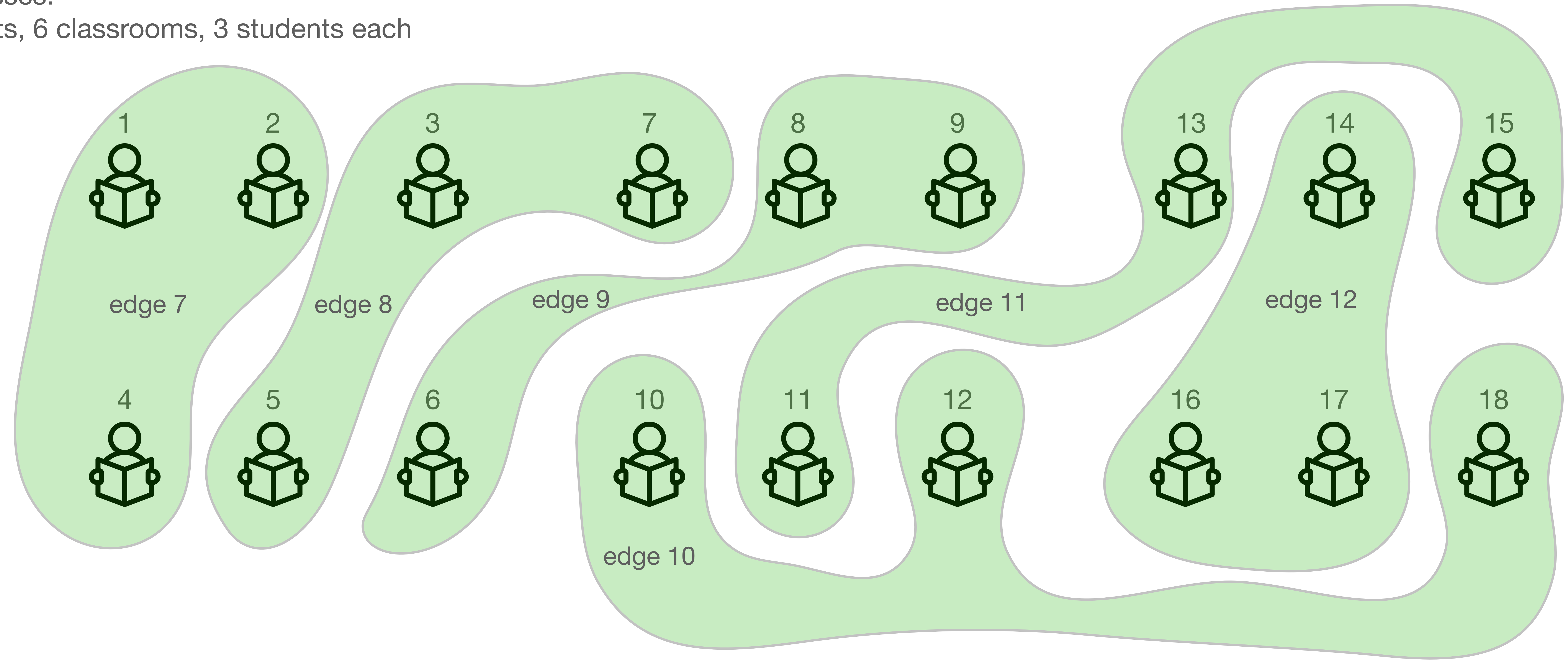
Morning classes:
n = 18 students, 6 classrooms, 3 students each



The high school problem

Construction

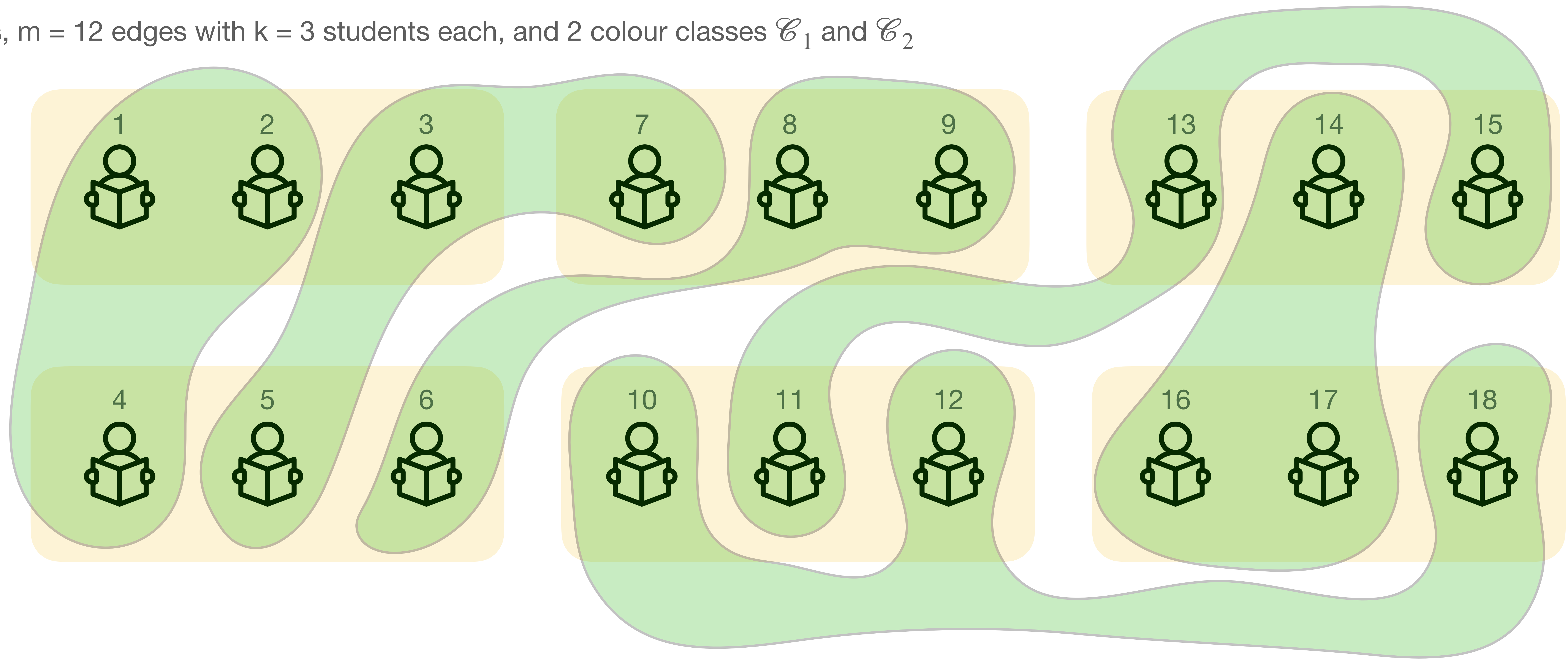
Afternoon classes:
n = 18 students, 6 classrooms, 3 students each



The high school problem

Construction

Total:
n = 18 vertices, m = 12 edges with k = 3 students each, and 2 colour classes $\mathscr{C}_1$ and $\mathscr{C}_2$



Overlapping edge construction

$\mathscr{C}_1$

edge 1	edge 2	edge 3	edge 4	edge 5	edge 6
1	1	1	0	0	0
1	0	0	1	1	0
0	1	0	1	0	1
0	0	1	0	1	1

$\mathscr{C}_2$

edge 7	edge 8	edge 9	edge 10	edge 11	edge 12
1	1	1	0	0	0
1	0	0	1	1	0
0	1	0	1	0	1
0	0	1	0	1	1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	0	0	1
0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	1
0	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	0

Overlapping edge construction

$\mathscr{C}_1$

edge 1	edge 2	edge 3	edge 4	edge 5	edge 6
1	1	1	0	0	0
1	0	0	1	1	0
0	1	0	1	0	1
0	0	1	0	1	1

$\mathscr{C}_2$

edge 7	edge 8	edge 9	edge 10	edge 11	edge 12
1	1	1	0	0	0
1	0	0	1	1	0
0	1	0	1	0	1
0	0	1	0	1	1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

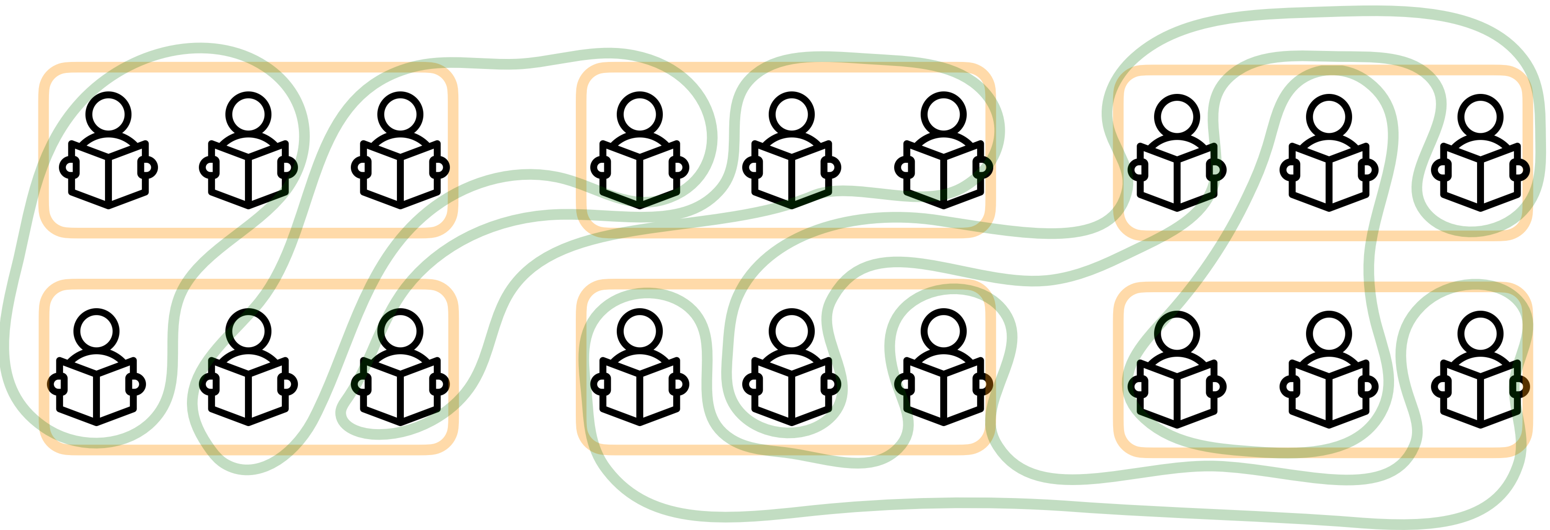
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	0	0	1
0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	1
0	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	0
1	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0
0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	1	1

Overlapping edge construction

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	0	0	1
0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	1
0	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	0
1	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0
0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	1	1

$$(\mathcal{S},1) - CFF(t,n)$$



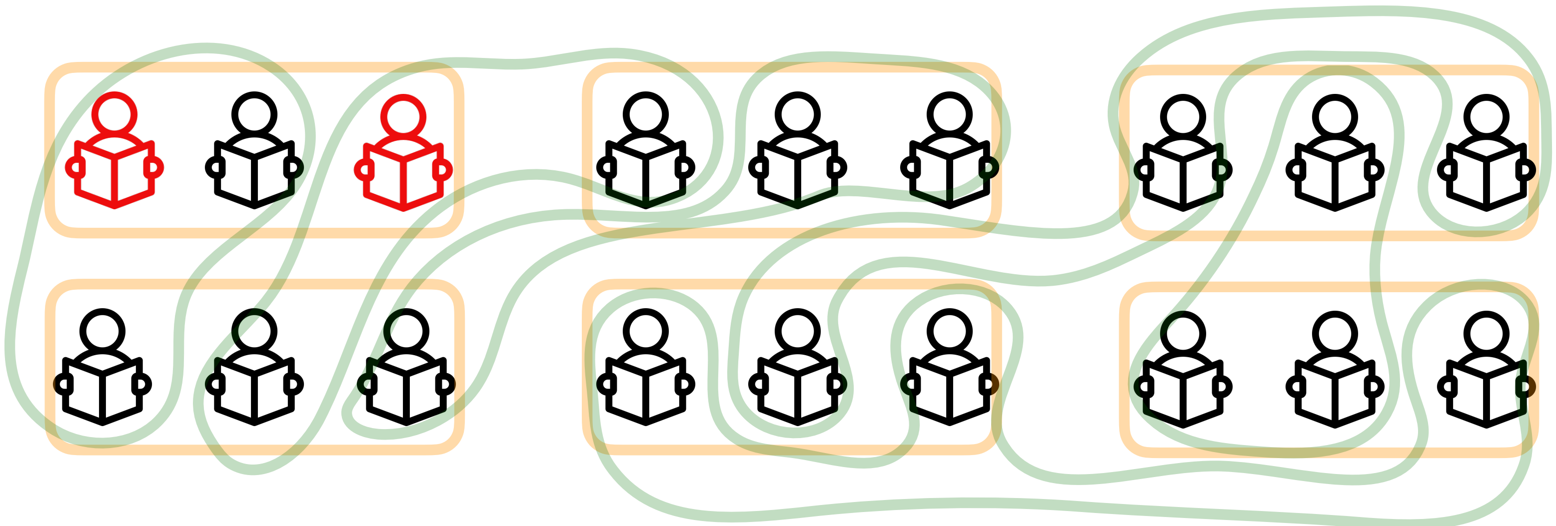
Overlapping edge construction

Edges 1, 7 and 8 are infected

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	0	0	1
0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	1
0	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	0
1	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0
0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	1	1

$(\mathcal{S},1) - CFF(t,n)$



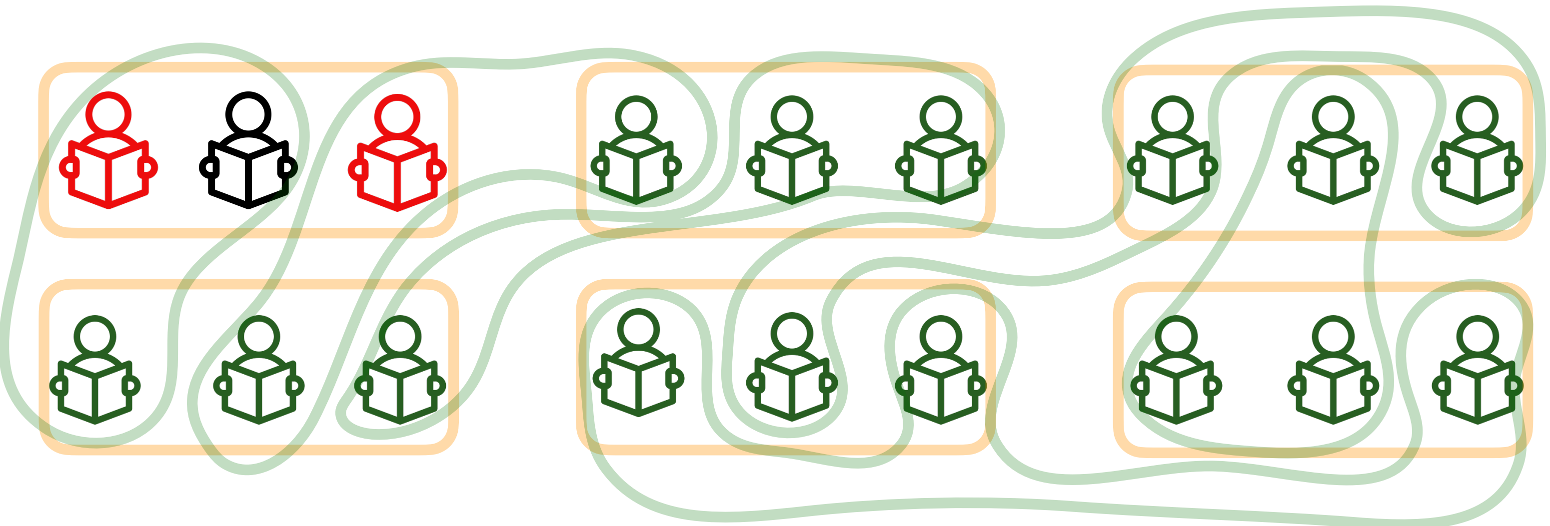
Overlapping edge construction

Edges 1, 7 and 8 are infected, students 4-18 are cleared out

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	0	0	1
0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	1
0	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	0
1	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0
0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	1	1

$$(\mathcal{S},1) - CFF(t,n)$$



Overlapping edge construction

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0
0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

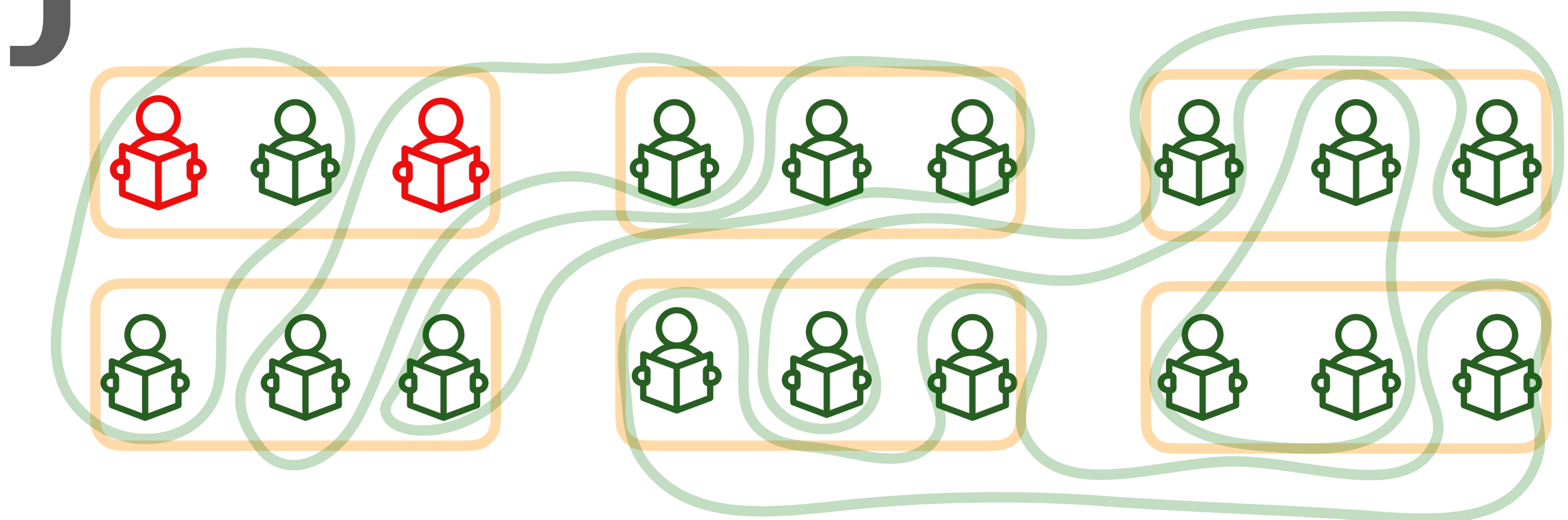
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	1	1	1	1	0	1	0	0	1
0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	1
0	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	0
1	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	1	0	0	0	1	1	0	0	1	0	0
0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	0	1	1

Edges 1, 7 and 8 are infected, students 4-18 are cleared out

Inside each infected edge, we can identify precisely the infected students

$$(\mathcal{S},1) - CFF(t = 2 \times (4+3), n = 18)$$

$$(\mathcal{S},1) - ECFF(t' = 2 \times 4, n = 18)$$



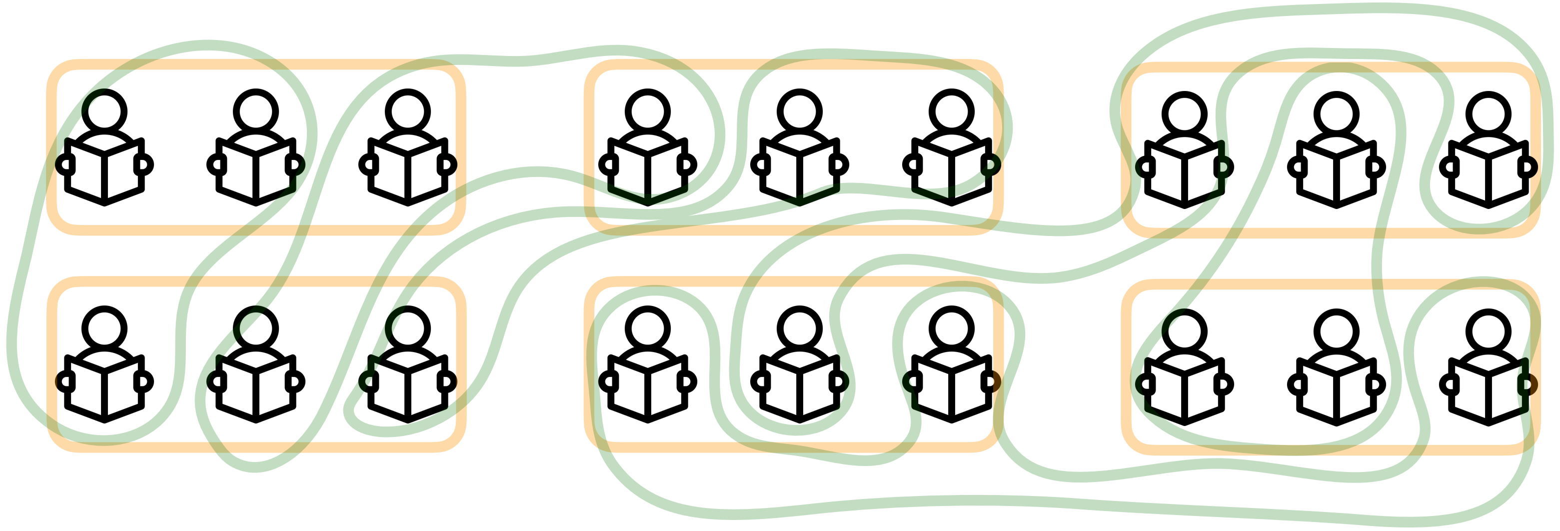
For a larger highschool

- **n = 900** studens
- Each student taking 4 courses (4 colour classes)
- Total of m = 120 courses (edges)
- Each course with 30 students (cardinality of edges)
- Tests:
  - Use $1 - CFF(7{,}30 = 120/4)$
  - t’ = 7x4 = 28 tests to detect infected edges (course of outbreak)
  - t = 28+30x4 = 148 tests to identify all infected individuals

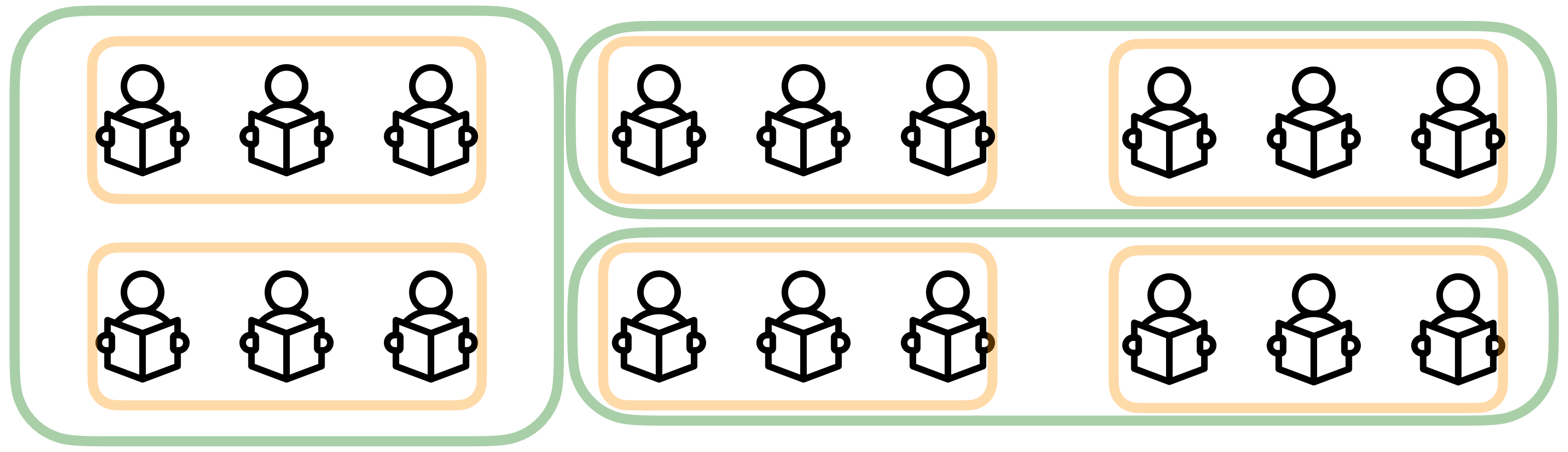
Overlapping edge construction

$(\mathcal{S},1) - CFF(t, n)$

- Consider a hypergraph $\mathcal{H}$ with edge chromatic number $\chi(\mathcal{H}) = \ell$ and colour classes $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$
- If $\mathcal{H}$ is **k-uniform**: we have $(\mathcal{S},1) - CFF(t, n)$ and $(\mathcal{S},1) - ECFF(t', n)$
  - Start with a $1 - CFF(t_1, n/k)$
  - $t \leq \ell \times (t_1 + k) \approx \ell \times (\log n/k + k)$
  - $t' \leq \ell \times t_1 \approx \ell \times \log n/k$



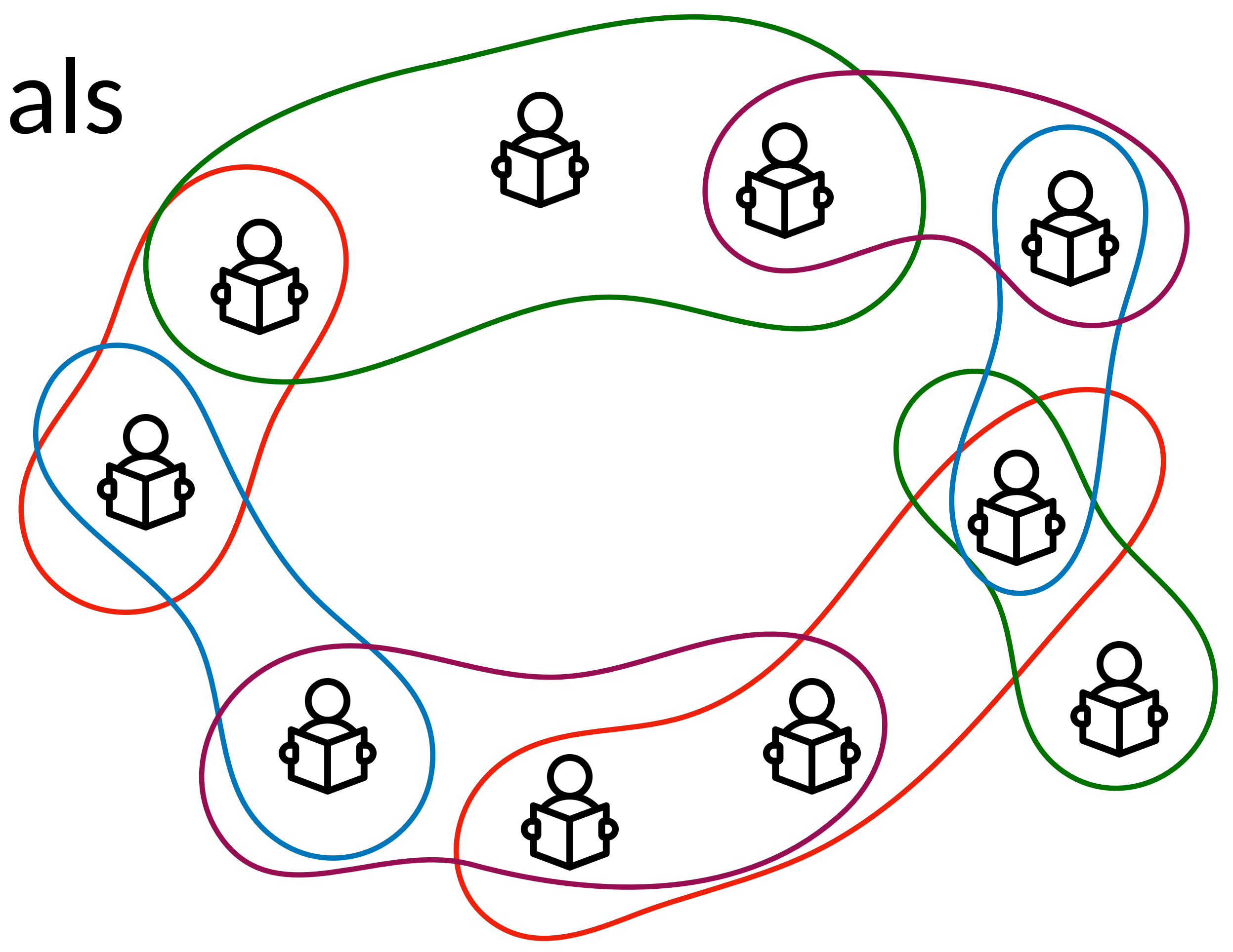
- If $\mathcal{H}$ has edges of **different cardinalities**, we have $(\mathcal{S},1) - CFF(t, n)$ and $(\mathcal{S},1) - ECFF(t', n)$
  - Start with $1 - CFF(t_i, |\mathcal{C}_i| + \delta_i), 1 \leq i \leq \ell$
  - $t = \sum_{i=1}^{\ell} (t_i + k_i)$, $\quad k_i =$ max edge in colour class $\mathcal{C}_i$
  - $t' = \sum_{i=1}^{\ell} t_i$



Overlapping edge construction

$(\mathcal{S}, r) - CFF(t, n)$

- Generalization for $(\mathcal{S}, r) - CFF(t, n)$ using **strong edge-colouring**
  - Assuming that r edges $\mathscr{E} = \{S_1, S_2, \ldots, S_r\}$ contain all infected individuals
  - There are at most r edges in $\mathscr{C}_i$ which intersect $\mathscr{E}$
    - $\mathscr{C}_i$ contains at most r infected edges
  - Use a combination of $r - CFF(t_i, |\mathscr{C}_i|)$ and $(r-1) - CFF(t_i', |\mathscr{C}_i|)$
  - $(\mathcal{S}, r) - CFF(t, n)$ with $t \leq \sum_{i=1}^{\ell} (t_i + k_i t_i')$, k_i = max edge in colour class $\mathscr{C}_i$



Future Work on structure-aware CFFs

- Explore other constraints of the applications
  - Limit on number of 1s per row
- Generalize definitions to allow flexible internal identification
  - Assume a bound on the number of infected items inside an edge (instead of edge size)
- Explore probabilistic constructions
- Compare constructions with known lower bounds

Thank you!

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